数理解析研究所講究録
601巻 1986年 11-18

Vorticity and viscosity

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060, JAPAN

This is a resume of my joint work with T. Miyakawa and H. Osada [36].

We consider the Navier-Stokes system

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad \nabla\cdot u = 0 \tag{1}$$

on the whole plane R^2, where u and p represents unknown velocity and pressure, respectively and $\nu > 0$ is the kinematic viscosity. Since the space dimension is two, the vorticity $v = \nabla \times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ is scalar. Moreover, v solves

$$\frac{\partial v}{\partial t} - \nu\Delta v + (u\cdot\nabla)v = 0 \tag{2a}$$

$$u(x,t) = \iint_{R^2} \nabla^{\perp}E(x-y)v(x,t)dx \tag{2b}$$

where $\nabla^{\perp} = (-\partial/\partial x_2, \partial/\partial x_1)$ and $E(x) = (2\pi)^{-1}\log|x|$. These equations are formally obtained by taking $\nabla\times$ of (1) and using the condition $\nabla\cdot u = 0$. As is well known the vorticity equation (2a)(2b) is formally equivalent to the Navier-Stokes system provided that u is assumed to decay to zero at space infinity.

We consider the initial value preblem for (1) or (2a)(2b) assuming only that initial vorticity $v(x,0)$ is a finite Radon measure. A typical example is N-point sources of votex, i.e.,

$$(3) \qquad v(x,0) = \textstyle\sum_{j=1}^{N}\alpha_j\delta(x-z_j).$$

Here z_j is a point on which j-th point source is located and α_j is a real number describing the strength of the source; δ is a Dirac measure supported at zero. One naive question is whether such point sources of vortex are smoothed out because of viscosity. In other words do solutions for (1) or (2a),(2b) exist globally-in-time and smooth for $t > 0$ even if $v(x,0)$ is a finite measure? When initial vorticity consists only one point source carried at zero (i.e. $N = 1$, $z_1 = 0$), we know an exact solution of (2a),(2b)

$$v(x,t) = \frac{\alpha_1}{4\pi\nu t}\exp\left(\frac{-|x|^2}{4\nu t}\right)$$

which is a constant multiple of the fundamental solution of the heat equation. For a general initial data we claim that a smooth solution exists globally in time. As anticipated, the viscosity smoothes singular vorticities.

Theorem ([36]). *Suppose that* $v(x,0)$ *is a finite Radon measure on* R^2. *Then there is a global solution* $v(x,t)$, $u(x,t)$ *to* (2a),(2b) *or* (1) *such that* v *and* u *are smooth for* $t > 0$ *and* $v(x,t)$ *converges to* $v(x,0)$ *under the weak topology of measures as* t *tends to zero*.

In [3] Benfatto, Esposito and Pulvirenti prove similar results under more stringent assumptions. They assume $v(x,0)$ is expressed by (3) and $|\alpha_j|$ is small compared with ν. Our results need no assumptions on particular forms or smallness of initial vorticity.

The main mathematical difficulty is that the initial energy on D

$$\iint_D |u(x,0)|^2\, dx$$

is not necessarily finite even if D is a bounded domain. If the initial energy is finite, it is classical that there is a global classical solutions to (1) (cf. [16,17,30]).

To construct such a solution we approximate initial vorticity by smooth functions and solve (2a),(2b) with approximate initial data. It is not difficult to construct a global solution for smooth data. We expect that solutions with approximate initial data converge to a true solution for the original problem. To carry out this process we need a priori estimates.

Lemma ([36]). *Suppose that* $v(x,0)$ *is smooth and*

$\iint_{R^2}|v(x,0)|dx \leq m$. *Let* $\Gamma_u(x,t;y,s)$ *is a fundamental solution to* (2a), *regarding* u *is a known function*. *Then*,

$$c(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{c(t-s)}\right) \leq \Gamma_u(x,t;y,s) \leq C(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{C(t-s)}\right)$$

with c *and* C > 0 *depending only on* m.

Estimates of fundamental solutions independent of the regularity of coefficients are obtained by Aronson [1] for linear parabolic equations of divergence form (see also [2]). Osada [25] extends the estimate for non-divengence form which includes (2a) as a typical example. The above a priori estimates enable us to carry out our original idea.

For uniqueness of the solution we do not know much. We show the uniqueness when v(x,0) is small. In particular, if v(x,0) is absolutely continuous with respect to Lebesgue measure, we can assert the uniqueness.

Our references include those of the paper [36] for the reader's convenience.

References

1. Aronson, D.G., Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73, 890-896 (1968).

2. Aronson, D.G., & J. Serrin, Local behavior of solutions of quasilinear parabolic equations. Arch. Rational Mech. Anal. 25. 81-122 (1967).

3. Benfatto, G., Esposito, R., & M. Pulvirenti, Planar Navier-Stokes flow for singular initial data. Nonlinear Anal. 9, 533-545 (1985).

4. Bergh, J., & J. Löfström, Interpolation Spaces, An Introduction. Berlin Heidelberg New York : Springer-Verlag 1976.

5. Brezis, H., & A. Friedman, Nonlinear parabolic equations involving measures as initial data. J. Math. Pures et appl. 62, 73-97 (1983).

6. Dobrushin, R.L., Prescribing a system of random variables by conditional distributions. Theory Prob. Appl. 15, 458-486 (1970).

7. Fabes, E.B., Jones, B.F., & N.M. Riviere, The initial value problem for the Navier-Stokes equations with data in L^p. Arch. Rational Mech. Anal. 45, 222-240 (1972).

8. Friedman, A., Partial Differential Equations of Parabolic Type. New Jersey : Prentice-Hall 1964.

9. Friedman, A., Partial Differential Equations. New York : Holt, Rinehart & Winston 1969.

10. Fujita, H., & T. Kato, On the Navier-Stokes initial value problem I. Arch. Rational Mech. Anal. 16, 269-315 (1964).

11. Giga, Y., & T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem. Arch. Rational Mech. Anal. 89, 267-281 (1985).

12. Giga, Y., Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. J. Differential Equations 62, 186-212 (1986).

13. Gilbarg, D., & N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed. Berlin Heidelberg New York :Springer-Verlag 1983.

14. Kato, T., Strong L^p-solutions of the Navier-Stokes equation in R^m,with applications to weak solutions. Math.Z. 187, 471-480 (1984).

15. Kato, T., Remarks on the Euler and Navier-Stokes equations in R^2.Nonlinear Functional Analysis and its Applications, F. E. Browder ed., Proc. of Symposia in Pure Math. 45, part 2, 1-8. Providence, RI: Amer. Math. Soc. 1986.

16. Ladyzhenskaya, O.A., The Mathematical Theory of Viscous Incompressible Flow. New York : Gordon & Breach 1969.

17. Leray, J., Etude de diverses équations intégrales non linéaires etde quelques problėms que pose l'hydrodynamique. J. Math. pures et appl., Serie 9, 12, 1-82 (1933).

18. Liu, T.-P., & M. Pierre, Source-solutions and asymptotic behavior in conservation laws. J. Differential Equations 51, 419-441 (1984).

19. Marchioro, C., & M. Pulvirenti, Hydrodynamics in two dimensions and vortex theory. Commun. Math. Phys. 84, 483-503 (1982).

20. Marchioro, C., & M. Pulvirenti, Euler evolution for singular initial data and vortex theory. Commun. Math. Phys. 91, 563-572 (1983).

21. McGrath, F.J., Nonstationary planar flow of viscous and ideal fluids. Arch. Rational Mech. Anal. 27, 329-348 (1968).

22. McKean Jr., H.P., Propagation of chaos for a class of nonlinear parabolic equations. Lecture series in diff. eq., Session 7 : Catholic Univ. 1967.

23. Niwa, Y., Semilinear heat equations with measures as initial data. preprint.

24. Osada, H., & S. Kotani, Propagation of chaos for the Burgers equation. J. Math. Soc. Japan 37, 275-294 (1985).

25. Osada, H., Diffusion processes with generators of generalized divergence form. J. Math. Kyoto Univ. to appear.

26. Osada, H., Propagation of chaos for the two dimensional Navier-Stokes equations. preprint ; Announcement : Proc. Japan Acad. 62,8-11 (1986).

27. Ponce, G., On two dimensional incompressible fluids. Commun.Partial Differ. Equations 11, 483-511 (1986).

28. Reed, M., & B. Simon, Methods of Modern Mathematical Physics Vol. I, II ; New York : Academic Press 1972, 1975.

29. Sznitman, A.S., Propagation of chaos result for the Burgers equation. Probab. Th. Rel. Fields 71, 581-613 (1986).

30. Temam, R., Navier-Stokes Equations. Amsterdam : North-Holland 1977.

31. Turkington, B., On the evolution of a concentrated vortex in an ideal fluid. preprint.

32. Wahl, W. von, The Equations of Navier-Stokes and Abstract Parabolic Equations. Braunschweig : Vieweg Verlag 1985.

33. Weissler, F.B., The Navier-Stokes initial value problem in L^p. Arch. Rational Mech. Anal. 74, 219-230 (1980).

34. Kato, T., & G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(R^2)$. preprint.

35. Baras, P., & M. Pierre, Problems paraboliques semi-linéaires avec données mesures. Applicable Analysis 18, 111-149 (1984).

36. Giga, Y., Miyakawa, T. & Osada, H., Two dimensional Navier-Stokes flow with measures as initial vorticity, preprint.